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ON UNIFORM f-VECTORS OF CUTSETS IN THE TRUNCATED BOOLEAN LATTICE

BÉLA BAJNOK and SHAHRIAR SHAHRIARI

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Let $[n]=\{1,2,\dots,n\}$ and let $2^{[n]}$ be the collection of all subsets of $[n]$ ordered by inclusion. $\mathcal{C}\subseteq 2^{[n]}$ is a *cutset* if it meets every maximal chain in $2^{[n]}$, and the *width* of $\mathcal{C}\subseteq 2^{[n]}$ is the minimum number of chains in a chain decomposition of $\mathcal{C}$. Fix $0\le m\le l\le n$. What is the smallest value of k such that there exists a cutset that consists only of subsets of sizes between m and l, and such that it contains exactly k subsets of size i for each $m\le i\le l$? The answer, which we denote by $g_n(m,l)$, gives a lower estimate for the width of a cutset between levels m and l in $2^{[n]}$. After using the Kruskal–Katona Theorem to give a general characterization of cutsets in terms of the number and sizes of their elements, we find lower and upper bounds (as well as some exact values) for $g_n(m,l)$.

1. Introduction

Let $2^{[n]}$ be the *Boolean lattice* of order n, that is the lattice of all subsets (often called *nodes*) of $[n]=\{1,2,\dots,n\}$ ordered by inclusion. For $0\le m\le n$ we define the m-th *level set* $\binom{[n]}{m}$ of $2^{[n]}$ as the set of all subsets of size m. The f-*vector* (or *profile*) $\mathbf{f}=(f_0,f_1,\dots,f_n)$ of a collection of subsets $\mathcal{A}\subseteq 2^{[n]}$ is defined by $f_m=|\mathcal{A}_m|$ where $\mathcal{A}_m=\mathcal{A}\cap\binom{[n]}{m}$ and $0\le m\le n$.

A collection of $l+1$ subsets $A_0\subset A_1\subset\cdots\subset A_l$ in $2^{[n]}$ is called a *chain* of length l. A *maximal chain* in $2^{[n]}$ is one that has length n. A collection of w nodes with the property that none of them contains another is called an *antichain* of size w. The *length* and the *width* of a collection of subsets $\mathcal{A}\subseteq 2^{[n]}$ are defined as the length of the longest chain and the size of the largest antichain in $\mathcal{A}$, respectively.

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A *cutset* in $2^{[n]}$ is defined as a collection of subsets $\mathcal{C} \subseteq 2^{[n]}$ which intersects all maximal chains. Trivially, every collection $\mathcal{C}$ which contains $\emptyset$ or $[n]$ is a cutset. In [3] we proved that for $n \geq 2$, the width of a cutset in the Boolean lattice of order n which does not contain $\emptyset$ or $[n]$ is greater than or equal to $n-1$, and that for $n \geq 3$ there exist cutsets of width $n-1$ in $2^{[n]}$. Thus, it is possible to construct a cutset in $2^{[n]}$ with f-vector $(0, \underbrace{n-1, n-1, \ldots, n-1}_{n-1}, 0)$. We then may ask for the smallest value of k for which there is a cutset in $2^{[n]}$ with f-vector $(0, \underbrace{k, k, \ldots, k}_{n-1}, 0)$. The original goal of our work was to show that this value is $n-2$ (see Corollary 3 below).

More generally, for $0 \leq m \leq l \leq n$ we define $g_n(m,l)$ to be the smallest value of k for which the $n+1$-tuple $(f_0, f_1, \ldots, f_n)$, defined by $f_i = k$ if $m \leq i \leq l$ and 0 otherwise, can be the f-vector of a cutset in $2^{[n]}$. Thus our goal above is then to find $g_n(1, n-1)$. Note that by symmetry we have $g_n(m,l) = g_n(n-l, n-m)$, so we may assume without loss of generality that $m \leq l \leq n-m$.

Before studying $g_n(m,l)$, we give a general characterization of f-vectors of cutsets in $2^{[n]}$. For a given profile $\mathbf{f} = (f_0, f_1, \ldots, f_n)$ and integer m_0 with $0 \leq m_0 \leq n$, we construct a *canonical* collection of subsets $\mathcal{C}(\mathbf{f}, m_0)$, with the property that there is a cutset in $2^{[n]}$ with profile $\mathbf{f}$ if and only if $\mathcal{C}(\mathbf{f}, m_0)$ is a cutset for some (or every) $0 \leq m_0 \leq n$. We then translate this qualitative criterion to a quantitative one: For a given $\mathbf{f} = (f_0, f_1, \ldots, f_n)$ and $0 \leq m_0 \leq n$, we describe an easily computable value $q(\mathbf{f}, m_0)$, so that $\mathbf{f}$ will be the profile of a cutset in $2^{[n]}$ exactly when $f_{m_0} \geq q(\mathbf{f}, m_0)$ for some (or every) $0 \leq m_0 \leq n$. These characterizations, which we present in Section 2, are essentially due to Daykin [7] (for a correction see [5] and then [4]), though we follow a treatment which is more suitable for our purposes.

We can then determine the values of $g_n(m,l)$ for $l \leq m+2$. Namely, we prove the following.

Theorem 1.1. *Let n be a positive integer.*

1. $g_n(m,m) = \binom{n}{m}$ *for every integer* $0 \leq m \leq n$.
2. $g_n(m,m+1) = \binom{n-1}{m}$ *for every integer* $0 \leq m \leq n-1$.
3. $g_n(m,m+2) = \sum_{j=0}^{m} \binom{n-2j-2}{m-j}$ *for every integer* $0 \leq m \leq n/2-1$.

Next, viewing m as fixed and $n \gg m$ (i.e., for all $n > n_0 = n_0(m)$), we develop upper and lower bounds for $g_n(m,l)$.

Theorem 1.2. *Suppose that m and n are non-negative integers and $n \gg m$. Then*

1. $\binom{n-2}{m} < g_n(m,l) \leq \sum_{j=0}^{m} \binom{n-2j-2}{m-j}$ *for every integer* $m+2 \leq l \leq n-m-1$, *and*
2. $\binom{n-3}{m} < g_n(m,n-m) \leq \sum_{j=0}^{m} \binom{n-2j-3}{m-j}$.

For $m=1$ we then get the following results.

Corollary 1.3. *Suppose that $n>4$ and $1 \leq l \leq n-1$ are integers. Then*

$$g_n(1,l) = \begin{cases} n & \text{if } l=1 \\ n-1 & \text{if } 2 \leq l \leq n-2 \\ n-2 & \text{if } l=n-1 \end{cases}$$

For $2 \leq m \ll n$, Theorems 1 and 2 give the "numerator" of the leading term of the m-binomial representation (see section 2) of $g_n(m,l)$. Namely, this value is equal to n if $l=m$, $n-1$ if $l=m+1$, $n-2$ if $m+2 \leq l \leq n-m-1$, and $n-3$ if $l=n-m$. It is striking that for a rather large range of values of l, $g_n(m,l)$ stays essentially unchanged.

We note that in Theorem 2, the ratio of the upper bound to the lower bound is approximately $1+\frac{m}{n}$, and thus the bounds are rather accurate as $n \gg m$.

Extremal problems regarding cutsets in the Boolean lattice have been the object of much study. For example see [8], [10], [12], [13], [15], [17], [18].

2. f-vectors of cutsets

Given a collection $\mathcal{B} \subseteq \binom{[n]}{m}$, the *shadow* and the *shade* of $\mathcal{B}$ will be denoted by $\triangle\mathcal{B}$ and $\triangledown\mathcal{B}$, respectively [1, Chapter 2], and are as usual defined by

$$\triangle\mathcal{B} = \left\{ A \in \binom{[n]}{m-1} \;\middle|\; A \subseteq B \text{ for some } B \in \mathcal{B} \right\},$$

$$\triangledown\mathcal{B} = \left\{ A \in \binom{[n]}{m+1} \;\middle|\; B \subseteq A \text{ for some } B \in \mathcal{B} \right\}.$$

We order the elements of $\binom{[n]}{m}$ by the *squashed order* (also called the *colex* order) [1, Chapter 7], that is for $A,B \in \binom{[n]}{m}$, we say $A <_S B$ if the largest element of the symmetric difference of A and B is in B. For $1 \leq K \leq \binom{n}{m}$, we define the *initial collection* $\mathcal{F}_m(K)$ and the *last collection* $\mathcal{L}_m(K)$ at level m as the first and last K elements in the squashed order at level m, respectively.

In addition, if $K \leq 0$, then $\mathcal{F}_m(K) = \mathcal{L}_m(K) = \emptyset$. The squashed order has the property that the shadow of an initial collection at level m is an initial collection at level $m-1$, and the shade of a last collection at level m is a last collection at level $m+1$. The Kruskal–Katona Theorem ([14], [16] or [1, Chapter 7]) states that the size of the shadow of K nodes at level m is greater than or equal to the size of the shadow of $\mathcal{F}_m(K)$ and, equivalently, the size of their shade is greater than or equal to the size of the shade of $\mathcal{L}_m(K)$.

Let Ω_n denote the set of $n+1$-tuples of integers $(a_0, a_1, \ldots, a_n)$ such that $0 \leq a_m \leq \binom{n}{m}$ for all $0 \leq m \leq n$. To see whether a given $\mathbf{f} \in \Omega_n$ is the profile of a cutset in $2^{[n]}$, we construct a collection of subsets $\mathcal{C} = \mathcal{C}(\mathbf{f}, m_0)$, called the *canonical* collection of subsets for profile $\mathbf{f}$ and for level m_0 $(0 \leq m_0 \leq n)$. As we show below, there is a cutset in $2^{[n]}$ with profile $\mathbf{f}$ if and only if this canonical collection is a cutset for some (or every) m_0.

Our construction is as follows. First we let $\mathcal{E}_0^{\uparrow} = \{\emptyset\}$, $\mathcal{C}_0^{\uparrow} = \mathcal{F}_0(f_0)$, and for $1 \leq m \leq n$ we recursively define $\mathcal{E}_m^{\uparrow} = \bigtriangledown(\mathcal{E}_{m-1}^{\uparrow} - \mathcal{C}_{m-1}^{\uparrow})$ and $\mathcal{C}_m^{\uparrow} = \mathcal{L}_m(|\mathcal{E}_m^{\uparrow}|) - \mathcal{L}_m(|\mathcal{E}_m^{\uparrow}| - f_m)$. Then $\mathcal{E}_m^{\uparrow}$ is a last collection at level m, and it is precisely the set of nodes from which there is a chain of length m to $\emptyset$ which is disjoint from $\mathcal{C}_i^{\uparrow}$ for all $0 \leq i \leq m-1$. Analogously, we let $\mathcal{E}_n^{\downarrow} = \{[n]\}$, $\mathcal{C}_n^{\downarrow} = \mathcal{L}_n(f_n)$, and for $0 \leq m \leq n-1$ we recursively define $\mathcal{E}_m^{\downarrow} = \bigtriangleup(\mathcal{E}_{m+1}^{\downarrow} - \mathcal{C}_{m+1}^{\downarrow})$ and $\mathcal{C}_m^{\downarrow} = \mathcal{F}_m(|\mathcal{E}_m^{\downarrow}|) - \mathcal{F}_m(|\mathcal{E}_m^{\downarrow}| - f_m)$. This time $\mathcal{E}_m^{\downarrow}$ is an initial collection at level m, and it is the set of nodes from which there is a chain of length $n-m$ to $[n]$ which is disjoint from $\mathcal{C}_i^{\uparrow}$ for all $m+1 \leq i \leq n$. Finally, we define $\mathcal{C} = \mathcal{C}(\mathbf{f}, m_0) = (\cup_{i=0}^{m_0} \mathcal{C}_i^{\uparrow}) \cup (\cup_{i=m_0+1}^{n} \mathcal{C}_i^{\downarrow})$.

We can easily see that $\mathcal{C} = \mathcal{C}(\mathbf{f}, m_0)$ is a cutset if and only if $\mathcal{E}_{m_0}^{\uparrow} \cap \mathcal{E}_{m_0}^{\downarrow} \subseteq \mathcal{C}_{m_0}$. Furthermore, the profile $(c_0, c_1, \ldots, c_n)$ of $\mathcal{C}$ satisfies $c_m \leq f_m$ for every m, and if $\mathcal{C}$ is not a cutset, then its profile is exactly $\mathbf{f}$.

For example, let $n = 5$. If $\mathbf{f} = (0, 2, 5, 6, 0, 0)$ then, for all $0 \leq m_0 \leq 5$, $\mathcal{C}(\mathbf{f}, m_0)$ becomes

$$\{\{1\}, \{2\}, \{1,3\}, \{2,3\}, \{1,4\}, \{2,4\}, \{3,4\}, \{1,2,5\}, \{1,3,5\}, \{2,3,5\}, \{1,4,5\}, \{2,4,5\}, \{3,4,5\}\}.$$

On the other, hand if $\mathbf{g} = (0, 2, 6, 5, 0, 0)$, then for $\mathcal{C}(\mathbf{g}, 5)$ we get

$$\{\{1\}, \{2\}, \{1,3\}, \{2,3\}, \{1,4\}, \{2,4\}, \{3,4\}, \{1,5\}, \{1,2,5\}, \{1,3,5\}, \{2,3,5\}, \{1,4,5\}, \{2,4,5\}\}.$$

It is easily seen that the first is a cutset, while the second one is a collection with profile $\mathbf{g}$ and not a cutset.

The next two propositions give us useful ways of determining whether a given vector $\mathbf{f}\in\Omega_n$ can be the profile of a cutset in $2^{[n]}$.

Proposition 2.4. *Let $\mathbf{f}\in\Omega_n$ and $0\le m_0\le n$. The canonical collection $\mathcal{C}(\mathbf{f},m_0)$ defined above is a cutset if and only if $|\mathcal{E}^{\uparrow}_{m_0}|+|\mathcal{E}^{\downarrow}_{m_0}|\le\binom{n}{m_0}+f_{m_0}$.*

Proof. If $\mathcal{C}$ is a cutset, then the assertion follows as

$$|\mathcal{E}^{\uparrow}_{m_0}|+|\mathcal{E}^{\downarrow}_{m_0}|=|\mathcal{E}^{\uparrow}_{m_0}\cap\mathcal{E}^{\downarrow}_{m_0}|+|\mathcal{E}^{\uparrow}_{m_0}\cup\mathcal{E}^{\downarrow}_{m_0}|\le|\mathcal{C}_{m_0}|+\binom{n}{m_0}\le f_{m_0}+\binom{n}{m_0}.$$

On the other hand, if $\mathcal{C}$ is not a cutset, then its profile must be exactly $\mathbf{f}$. Furthermore, as $\mathcal{E}^{\downarrow}_{m_0}$ is an initial segment and $\mathcal{E}^{\uparrow}_{m_0}$ is a last segment at level m_0, their intersection has size greater than f_{m_0} and their union is all of $\binom{[n]}{m_0}$. Therefore, in this case we have $|\mathcal{E}^{\uparrow}_{m_0}|+|\mathcal{E}^{\downarrow}_{m_0}|=|\mathcal{E}^{\uparrow}_{m_0}\cap\mathcal{E}^{\downarrow}_{m_0}|+|\mathcal{E}^{\uparrow}_{m_0}\cup\mathcal{E}^{\downarrow}_{m_0}|>f_{m_0}+\binom{n}{m_0}$. ∎

The next proposition shows the importance of the canonical collection. It further shows that the choice of m_0 is immaterial.

Proposition 2.5. *Let $\mathbf{f}\in\Omega_n$. The following are equivalent.*

1. *$\mathbf{f}$ is the profile of a cutset in $2^{[n]}$.*
2. *$\mathcal{C}(\mathbf{f},m_0)$ is a cutset for some $0\le m_0\le n$.*
3. *$\mathcal{C}(\mathbf{f},m_0)$ is a cutset for every $0\le m_0\le n$.*

Proof. Clearly, if $\mathcal{C}(\mathbf{f},m_0)$ is a cutset for some $0\le m_0\le n$, then there is a cutset in $2^{[n]}$ with profile $\mathbf{f}$. Therefore, it is enough to prove that if there is an $\mathcal{A}\subseteq 2^{[n]}$ which is a cutset with profile $\mathbf{f}$, then $\mathcal{C}(\mathbf{f},m_0)$ is a cutset for every $0\le m_0\le n$.

For $m=0,\dots n$, let $\mathcal{A}_m=\{A\in\mathcal{A}\mid |A|=m\}$. We define $\mathcal{B}^{\uparrow}_m$ and $\mathcal{B}^{\downarrow}_m$ recursively by $\mathcal{B}^{\uparrow}_0=\{\emptyset\},\mathcal{B}^{\uparrow}_m=\triangledown(\mathcal{B}^{\uparrow}_{m-1}-\mathcal{A}_{m-1})$ for $m=1,2,\dots,n$, and $\mathcal{B}^{\downarrow}_n=\{[n]\},\mathcal{B}^{\downarrow}_m=\triangle(\mathcal{B}^{\downarrow}_{m+1}-\mathcal{A}_{m+1})$ for $m=0,1,\dots,n-1$. Since $\mathcal{A}$ is a cutset, we must have $\mathcal{B}^{\downarrow}_m\cap\mathcal{B}^{\uparrow}_m\subseteq\mathcal{A}_m$ for every $0\le m\le n$.

Keeping the notations established prior to Proposition 4, we now use downward induction on m to prove that $|\mathcal{B}^{\downarrow}_m|\ge|\mathcal{E}^{\downarrow}_m|$ and $|\mathcal{B}^{\uparrow}_m|\ge|\mathcal{E}^{\uparrow}_m|$ for every $0\le m\le n$. Clearly $|\mathcal{B}^{\downarrow}_n|\ge|\mathcal{E}^{\downarrow}_n|$. Using, in order, the definition of $\mathcal{B}^{\downarrow}_m$, the Kruskal–Katona theorem, the triangle inequality, the inductive hypothesis, the fact that $\mathcal{C}^{\downarrow}_{m+1}\subseteq\mathcal{E}^{\downarrow}_{m+1}$, that $\mathcal{E}^{\downarrow}_{m+1}-\mathcal{C}^{\downarrow}_{m+1}$ is an initial segment at level $m+1$, and the definition of $\mathcal{E}^{\downarrow}_m$, we can write

$$\begin{aligned}
|\mathcal{B}^{\downarrow}_{m}| &= |\triangle(\mathcal{B}^{\downarrow}_{m+1} - \mathcal{A}_{m+1})| \\
&\geq |\triangle \mathcal{F}_{m+1}(|\mathcal{B}^{\downarrow}_{m+1} - \mathcal{A}_{m+1}|)| \\
&\geq |\triangle \mathcal{F}_{m+1}(|\mathcal{B}^{\downarrow}_{m+1}| - |\mathcal{A}_{m+1}|)| \\
&\geq |\triangle \mathcal{F}_{m+1}(|\mathcal{E}^{\downarrow}_{m+1}| - |\mathcal{C}^{\downarrow}_{m+1}|)| \\
&= |\triangle \mathcal{F}_{m+1}(|\mathcal{E}^{\downarrow}_{m+1} - \mathcal{C}^{\downarrow}_{m+1}|)| \\
&= |\triangle(\mathcal{E}^{\downarrow}_{m+1} - \mathcal{C}^{\downarrow}_{m+1})| \\
&= |\mathcal{E}^{\downarrow}_{m}|,
\end{aligned}$$

as claimed. The assertion for $|\mathcal{B}^{\uparrow}_{m}|$ can be proved similarly.

Our assertion now follows from Proposition 4, as

$$\begin{aligned}
|\mathcal{E}^{\uparrow}_{m_0}| + |\mathcal{E}^{\downarrow}_{m_0}| \leq |\mathcal{B}^{\uparrow}_{m_0}| + |\mathcal{B}^{\downarrow}_{m_0}| &= |\mathcal{B}^{\uparrow}_{m_0} \cap \mathcal{B}^{\downarrow}_{m_0}| + |\mathcal{B}^{\uparrow}_{m_0} \cup \mathcal{B}^{\downarrow}_{m_0}| \\
&\leq |\mathcal{A}_{m_0}| + |\mathcal{B}^{\uparrow}_{m_0} \cup \mathcal{B}^{\downarrow}_{m_0}| \leq f_{m_0} + \binom{n}{m_0}.
\end{aligned}$$

∎

A quantitative version of Proposition 5 can be formulated as follows.

Given positive integers K and m, there exist unique integers $a_m > a_{m-1} > \cdots > a_t \geq t \geq 1$ such that

$$K = \binom{a_m}{m} + \binom{a_{m-1}}{m-1} + \cdots + \binom{a_t}{t}.$$

This is called the *m-binomial representation* of K [1, Theorem 7.2.1]. Using the m-binomial representation of K it is easy to describe the set numbered K in the squashed order on the m-th level of the Boolean lattice [1, page 117].

For each positive integer m we define a (boundary) operator ∂_m [11] ([6] and [9] have other notations) on the integers as follows: If K is a positive integer with an m-binomial representation as above then

$$\partial_m(K) = \binom{a_m}{m-1} + \binom{a_{m-1}}{m-2} + \cdots + \binom{a_t}{t-1},$$

and for non-positive K set $\partial_m(K) = 0$. Note that ∂_m is weakly increasing.

With this operator we can write

$$\triangle \mathcal{F}_m(K) = \mathcal{F}_{m-1}(\partial_m(K)), \qquad \text{and} \qquad \triangledown \mathcal{L}_m(K) = \mathcal{L}_{m+1}(\partial_{n-m}(K)),$$

and the Kruskal–Katona theorem becomes $|\triangle \mathcal{B}| \geq \partial_m(|\mathcal{B}|)$ and $|\triangledown \mathcal{B}| \geq \partial_{n-m}(|\mathcal{B}|)$ where B is a collection of subsets of $[n]$ at level m.

Theorem 2.6. *For a given* $\mathbf{f}=(f_0,f_1,\dots,f_n)\in\Omega_n$, *define*

$$\mathbf{u}(\mathbf{f})=(u_0,u_1,\dots,u_n) \qquad \text{and} \qquad \mathbf{v}(\mathbf{f})=(v_0,v_1,\dots,v_n)$$

by:

$u_0=1$ *and* $u_{m+1}=\partial_{n-m}(u_m-f_m)$ *for* $m=0,\dots,n-1$,
$v_n=1$ *and* $v_{m-1}=\partial_m(v_m-f_m)$ *for* $m=1,\dots,n$.

Then the following are equivalent.

1. $\mathbf{f}$ *is the profile of a cutset in* $2^{[n]}$.
2. $u_m+v_m-f_m\le\binom{n}{m}$ *for some* $0\le m\le n$.
3. $u_m+v_m-f_m\le\binom{n}{m}$ *for every* $0\le m\le n$.

Proof. $(u_0,u_1,\dots,u_n)$ and $(v_0,v_1,\dots,v_n)$ were chosen so that $|\mathcal{E}_m^{\uparrow}|=u_m$ and $|\mathcal{E}_m^{\downarrow}|=v_m$, so the statements follow from Propositions 4 and 5. ∎

Note that, as a special case when $f_0=f_n=0$, $\mathbf{f}$ is the profile of a cutset in $2^{[n]}$ if and only if $v_0=0$ (or if and only if $u_n=0$).

Remark. We can give a quantitative description of the canonical collection $\mathcal{C}(\mathbf{f},m_0)=(\cup_{i=0}^{m_0}\mathcal{C}_i^{\uparrow})\cup(\cup_{i=m_0+1}^{n}\mathcal{C}_i^{\downarrow})$ as follows. If levels of $2^{[n]}$ are in the squashed order, then $\mathcal{C}_i^{\uparrow}$ is the segment at level i starting with node numbered $\binom{n}{i}-u_i+1$ and ending with $\min\{\binom{n}{i}-u_i+f_i,\binom{n}{i}\}$; $\mathcal{C}_i^{\downarrow}$ is the segment at level i starting with node numbered $\max\{1,v_i-f_i+1\}$ and ending with v_i.

3. Exact values: Proof of Theorem 1

We will use Theorem 6 to determine the values of $g_n(m,l)$ for $l=m$, $m+1$, and $m+2$.

The case $l=m$ is obvious. For $l=m+1$, we can easily see that the canonical collection with profile $\mathbf{f}=(0,0,\dots,0,\binom{n-1}{m},\binom{n-1}{m},0,\dots,0)$ (where the nonzero components occur at levels m and $m+1$) consists of subsets of size m that do not contain n, together with subsets of size $m+1$ that contain n. This collection is a cutset. On the other hand, the canonical collection with profile $\mathbf{f}'=(0,0,\dots,0,\binom{n-1}{m},\binom{n-1}{m}-1,0,\dots,0)$ is not a cutset as there exists a maximal chain through the node $\{n-m,n-m+1,\dots,n\}$ which does not intersect the collection.

For the case $l=m+2$, the result is trivial for $m=0$ and hence assume that $m\ge1$. Define $f=\sum_{j=0}^{m-1}\binom{n-2j-2}{m-j}$, and we need to prove that $g_n(m,l)=f+1$. We will show that $\mathbf{f}=(0,0,\dots,0,f+1,f,f,0,\dots,0)$ (where the nonzero components occur at levels m, $m+1$, and $m+2$) is the profile of a cutset, but that $\mathbf{f}'=(0,0,\ldots,0,f,f,f,0,\ldots,0)$ is not the profile of a cutset.

Before starting the proof we need to establish a binomial identity which establishes a relationship between two *vertical* columns in the arithmetic (a.k.a. Pascal's) triangle. Let n be a positive integer, let $1\leq m\leq n/2$, and let $0\leq d\leq m-1$. Using downward induction on d (with base case $d=m-1$) it can be easily shown that

$$\sum_{j=0}^{m-d-1}\binom{n-2j-2}{m-j}+\sum_{j=0}^{m-d-1}\binom{n-2j-1}{m+2-j}+\binom{n-2m+2d}{d+2}=\binom{n}{m+2}.$$

In our computations below we rely on the case $d=0$ of this identity, namely:

$$f+\sum_{j=0}^{m-1}\binom{n-2j-1}{m+2-j}+\binom{n-2m}{2}=\binom{n}{m+2}.$$

We now use Theorem 6 to show that $\mathbf{f}$ is the profile of a cutset. Computing $\mathbf{v}(\mathbf{f})$ we get

$$\begin{aligned}
v_i &= \binom{n}{i} \qquad\qquad \text{for} \quad i=n,n-1,\ldots,m+2,\\
v_{m+1} &= \partial_{m+2}\left[\binom{n}{m+2}-f\right]\\
&= \partial_{m+2}\left[\sum_{j=0}^{m-1}\binom{n-2j-1}{m+2-j}+\binom{n-2m}{2}\right]\\
&= \sum_{j=0}^{m-1}\binom{n-2j-1}{m+1-j}+\binom{n-2m}{1},\\
v_m &= \partial_{m+1}\left[\sum_{j=0}^{m-1}\binom{n-2j-1}{m+1-j}+\binom{n-2m}{1}-f\right]\\
&= \partial_{m+1}\left[\sum_{j=0}^{m-1}\binom{n-2j-2}{m+1-j}+\binom{n-2m}{1}\right]\\
&= \sum_{j=0}^{m-1}\binom{n-2j-2}{m-j}+\binom{n-2m}{0}\\
&= f+1,\\
v_{m-1} &= \partial_m[f+1-f-1]\\
&= 0, \qquad\qquad \text{and}
\end{aligned}$$

$$v_i = 0 \qquad \text{for} \quad i = m-1, m-2, \ldots, 0,$$

hence $\mathbf{f}$ is the profile of a cutset. As $\mathbf{f}'$ differs from $\mathbf{f}$ only in its m-th coordinate, we still have $v_m = f+1$. Continuing, we get

$$\begin{aligned} v_{m-1} &= \partial_m[f+1-f] \\ &= \partial_m\left[\binom{m}{m}\right] \\ &= \binom{m}{m-1}, \qquad \text{and} \\ v_i &= \binom{m}{i} \qquad \text{for} \quad i = m-1, m-2, \ldots, 0. \end{aligned}$$

In particular, $v_0 = 1$, so $\mathbf{f}'$ is not the profile of a cutset. ∎

Remark. We can also construct a (non-canonical) cutset with f-vector $\mathbf{f} = (0,0,\ldots,0,f{+}1,f,f,0,\ldots,0)$ where $f = \sum_{j=0}^{m-1}\binom{n-2j-2}{m-j}$. We define $\mathcal{Q}_0 = \binom{[n-2]}{m}$ and for $j = 1,2,\ldots,m$ we let

$$\begin{aligned} \mathcal{Q}_j &= \left\{ A \cup \{n-1, n-3, \ldots, n-2j+1\} | A \in \binom{[n-2j-2]}{m-j} \right\}, \\ \mathcal{R}_j &= \{Q \cup \{n-2j+2\} | Q \in \mathcal{Q}_{j-1}\}, \quad \text{and} \\ \mathcal{S}_j &= \{R \cup \{n-2j+1\} | R \in \mathcal{R}_{j-1}\}. \end{aligned}$$

Then it is rather straight forward to check that $\mathcal{Q} = \bigcup_{j=0}^m \mathcal{Q}_j$, $\mathcal{R} = \bigcup_{j=1}^m \mathcal{R}_j$, and $\mathcal{S} = \bigcup_{j=1}^m \mathcal{S}_j$ are collections of nodes at levels m, $m+1$, and $m{+}2$, respectively, $|\mathcal{Q}| = f{+}1$, $|\mathcal{R}| = f$, and $|\mathcal{S}| = f$, and that $\mathcal{Q}\cup\mathcal{R}\cup\mathcal{S}$ is a cutset, thus providing a cutset with f-vector $\mathbf{f} = (0,0,\ldots,0,f+1,f,f,0,\ldots,0)$. We omit the details but only briefly sketch the idea behind the construction: Note that $\mathcal{Q}_0$ consists of *all* subsets of size m that do not contain n or $n{-}1$. Likewise, $\mathcal{R}_1$ consists of all subsets of size $m{+}1$ that contain n but not $n{-}1$ and $\mathcal{S}_1$ consists of all subsets of size $m{+}2$ that contain n and $n{-}1$. Thus any maximal chain in $2^{[n]}$ that does not intersect $\mathcal{Q}_0\cup\mathcal{R}_1\cup\mathcal{S}_1$ will have subsets of size m, $m{+}1$ and $m{+}2$ that contain $n{-}1$ but not n. Now the poset of subsets of $[n]$ that contain $n-1$ but not n is isomorphic (as a poset) to $2^{[n-2]}$. We now restrict ourself to this poset and continue the construction recursively.

4. Bounds: Proof of Theorem 2

We need the following propositions.

Proposition 4.7. *For $m \le l < n/2$ we have $g_n(m, n-m) \le g_{n-1}(m, l)$.*

Proof. Let $\mathcal{A} \subseteq 2^{[n-1]}$ be a cutset with f-vector $(0,\dots,0,g,\dots,g,0,\dots,0)$, where $g = g_{n-1}(m,l)$ and the nonzero entries are at levels $m \le i \le l$. Let $\overline{\mathcal{A}} = \{[n] \setminus A | A \in \mathcal{A}\}$. Note that by symmetry $\overline{\mathcal{A}}$ is a cutset in the poset $\mathcal{Q}$ consisting of all subsets of $[n]$ that contain n. Now define $\mathcal{B} = \mathcal{A} \cup \overline{\mathcal{A}}$. Note that since $l < n-l$, $\mathcal{B}$ has f-vector $(0,\dots,0,g,\dots,g,0,\dots,0,g,\dots,g,0,\dots,0)$, where the nonzero entries are at levels $m \le i \le l$ and $n-l \le i \le n-m$. It now suffices to show that $\mathcal{B}$ is a cutset in $2^{[n]}$.

Let $\mathcal{C} = C_0 \subset C_1 \subset \cdots \subset C_n$ be a maximal chain in $2^{[n]}$ with $|C_i| = i$ for $0 \le i \le n$, and suppose, indirectly, that $\mathcal{B} \cap \mathcal{C} = \emptyset$. Since $\mathcal{A}$ is a cutset in $2^{[n-1]}$ with support between levels m and l, we must have $n \in C_k$ for every $l \le k \le n$. Similarly, since $\overline{\mathcal{A}}$ is a cutset in $\mathcal{Q}$, we must have $n \notin C_k$ for every $0 \le k \le n-l$. This can only happen if $n-l < l$ which is a contradiction. ∎

Proposition 4.8. *Suppose that $m+2 \le l \le n-m-1$ and $g_{n-1}(m+1, l-1) > \binom{n-2}{m}$. Then $g_n(m,l) > \binom{n-2}{m}$.*

Proof. Let $\mathbf{f} = (0,0,\dots,0,\binom{n-2}{m},\binom{n-2}{m},\dots,\binom{n-2}{m},0,\dots,0)$, where the nonzero entries occur between levels m and l. Suppose, indirectly, that $\mathbf{f}$ is the profile of a cutset in $2^{[n]}$. Then, by Proposition 5, the canonical collection $\mathcal{C}(\mathbf{f}, n)$ is such a cutset. If $C_i = \mathcal{C} \cap \binom{[n]}{i}$ are the level sets of the collection, then it is easy to see that $C_m = \binom{[n-2]}{m}$ and $C_{m+1} = \{A \cup \{n-1\} | A \in \binom{[n-2]}{m}\}$. Therefore $\cup_{i=m+2}^{l} C_i$ is a (canonical) cutset in $\{A \cup \{n\} | A \subseteq [n-1]\}$ of profile $(0,0,\dots,0,\binom{n-2}{m},\binom{n-2}{m},\dots,\binom{n-2}{m},0,\dots,0)$, where the nonzero entries occur between levels $m+2$ and l. But $\{A \cup \{n\} | A \subseteq [n-1]\}$ is isomorphic to $2^{[n-1]}$, so we must have $g_{n-1}(m+1, l-1) \le \binom{n-2}{m}$, a contradiction. ∎

Proposition 4.9. *Define $\mathbf{f} = (f_0, f_1, \dots, f_n)$ as follows. Let $f_m = f_{m+1} = f_{n-m-1} = f_{n-m} = \binom{n-3}{m}$, $f_i = f_{n-i} = \binom{n+m-i-1}{m+1}$ for $i = m+2, m+3, \dots, \lfloor n/2 \rfloor$, and $f_i = 0$ otherwise. Then $\mathbf{f}$ is not the profile of a cutset in $2^{[n]}$.*

Proof. For $\mathbf{v}(\mathbf{f})$ we get the following.

$$v_i = \binom{n}{i} \qquad \text{for} \quad i = n, n-1, \dots, n-m,$$

$$\begin{aligned}
v_{n-m-1} &= \partial_{n-m}\left[\binom{n}{n-m} - \binom{n-3}{n-m-3}\right] \\
&= \partial_{n-m}\left[\binom{n-1}{n-m} + \binom{n-2}{n-m-1} + \binom{n-3}{n-m-2}\right] \\
&= \binom{n-1}{n-m-1} + \binom{n-2}{n-m-2} + \binom{n-3}{n-m-3}, \\
v_{n-m-2} &= \partial_{n-m-1}\left[\binom{n-1}{n-m-1} + \binom{n-2}{n-m-2}\right. \\
&\qquad \left. + \binom{n-3}{n-m-3} - \binom{n-3}{n-m-3}\right] \\
&= \partial_{n-m-1}\left[\binom{n-1}{n-m-1} + \binom{n-2}{n-m-2}\right] \\
&= \binom{n-1}{n-m-2} + \binom{n-2}{n-m-3}, \\
v_{n-m-3} &= \partial_{n-m-2}\left[\binom{n-1}{n-m-2} + \binom{n-2}{n-m-3} - \binom{n-3}{n-m-4}\right] \\
&= \partial_{n-m-2}\left[\binom{n-1}{n-m-2} + \binom{n-3}{n-m-3}\right] \\
&= \binom{n-1}{n-m-3} + \binom{n-3}{n-m-4}, \\
v_{n-m-4} &= \partial_{n-m-3}\left[\binom{n-1}{n-m-3} + \binom{n-3}{n-m-4} - \binom{n-4}{n-m-5}\right] \\
&= \partial_{n-m-3}\left[\binom{n-1}{n-m-3} + \binom{n-4}{n-m-4}\right] \\
&= \binom{n-1}{n-m-4} + \binom{n-4}{n-m-5},
\end{aligned}$$

and continuing as above, we get $v_k = \binom{n-1}{k} + \binom{k+m}{k-1}$ for every $k \geq \lfloor (n-1)/2 \rfloor$.

Now note that since $\mathbf{f}$ is symmetrical, for $\mathbf{v}(\mathbf{f})$ and $\mathbf{u}(\mathbf{f})$ we have $v_i = u_{n-i}$ for every $0 \leq i \leq n$. Therefore

$$\begin{aligned}
v_{\lfloor n/2 \rfloor} + u_{\lfloor n/2 \rfloor} &= v_{\lfloor n/2 \rfloor} + v_{\lceil n/2 \rceil} \\
&= \binom{n-1}{\lfloor n/2 \rfloor} + \binom{\lfloor n/2 \rfloor + m}{\lfloor n/2 \rfloor - 1} + \binom{n-1}{\lceil n/2 \rceil} + \binom{\lceil n/2 \rceil + m}{\lceil n/2 \rceil - 1}
\end{aligned}$$

$$= \binom{n}{\lfloor n/2 \rfloor} + f_{\lfloor n/2 \rfloor} + f_{\lceil n/2 \rceil},$$

thus $\mathbf{f}$ cannot be the profile of a cutset by Theorem 6. ∎

Proof of Theorem 2 and Corollary 3. The upper bound in 1 follows from Theorem 1 part 3 since $g_n(m,l)$ is weakly decreasing with l. The upper bound in 2 is a consequence of Proposition 7 (take $l = m+2$ assuming $n > 2m+4$) and Theorem 1 part 3. Proposition 9 implies that $g_n(m,n-m) > \min\{\binom{n-3}{m}, \binom{\lfloor n/2 \rfloor + m-1}{m+1}\}$. It is an elementary exercise to check that this minimum is $\binom{n-3}{m}$ for $n \gg m$ (e.g. if $n \geq 3^{m+1}(m+1)$), establishing the lower bound in 2. Finally, the lower bound in 1 is a consequence of Proposition 8 taking $l = n-m-1$, and noting that $\binom{n-4}{m+1} \geq \binom{n-2}{m}$ for $n \gg m$ (e.g. if $n \geq 8(m+1)$) and the lower bound in 2.

The cases of $l = 1,2$ (and 3) of Corollary 3 follow from Theorem 1. The assertions for $3 \leq l \leq n-1$ follow from Theorem 2 if $n \geq 18$ (see above). The cases $5 \leq n \leq 17$ were checked directly using Theorem 6 (and a simple computer program). ∎

We close this section by proving a partial complement to Proposition 7.

Proposition 4.10. *For $n/2 - 1 < l \leq n-m-2$ we have $g_n(m,n-m) \geq g_{n-1}(m,l)$.*

Proof. Consider the canonical collection $\mathcal{C} = \mathcal{C}(\mathbf{f}, l)$ where $\mathbf{f} = (f_0, f_1, \ldots, f_n)$ is defined by $f_i = g_n(m,n-m)$ for $m \leq i \leq n-m$ and 0 otherwise. Then $\mathcal{C}$ is a cutset in $2^{[n]}$. Write $\mathcal{D}_1 = \{C \in \mathcal{C} \mid |C| \leq l\}$ and $\mathcal{D}_2 = \{C \in \mathcal{C} \mid |C| \geq l+1\}$.

Suppose, indirectly, that $g_n(m,n-m) < g_{n-1}(m,l)$. Then $\mathcal{D}_1$ is not a cutset in $2^{[n-1]}$, in particular, $\mathcal{D}_1 \subseteq 2^{[n-1]}$. Since $l \geq n-1-l$, by symmetry $\mathcal{D}_2$ is disjoint from $2^{[n-1]}$. Therefore $\mathcal{D}_1 \cup \mathcal{D}_2 = \mathcal{C}$ is not a cutset in $2^{[n]}$, a contradiction. ∎

5. Examples and conjectures

The following table has the exact values of $g_n(m,l)$ for $n = 100$, $m = 4$ and every $4 \leq l \leq 96$. These were found using Theorem 6. We have also given the 4-binomial representation of $g_n(m,l)$ in the third column.

l	$g_{100}(4,l)$	4-binomial representation of $g_{100}(4,l)$
4	$3,921,225$	$\binom{100}{4}$
5	$3,764,376$	$\binom{99}{4}$
6	$3,759,624$	$\binom{98}{4}+\binom{96}{3}+\binom{94}{2}+\binom{93}{1}$
7	$3,759,526$	$\binom{98}{4}+\binom{96}{3}+\binom{93}{2}+\binom{88}{1}$
$8\leq l\leq 95$	$3,759,525$	$\binom{98}{4}+\binom{96}{3}+\binom{93}{2}+\binom{87}{1}=\binom{100}{4}-\binom{100}{3}$
96	$3,607,527$	$\binom{97}{4}+\binom{95}{3}+\binom{92}{2}+\binom{86}{1}=\binom{99}{4}-\binom{99}{3}$

Theorem 1 gives the exact value for $4\leq l\leq 6$. For $6\leq l\leq 95$, Theorem 2 gives $3,612,280=\binom{98}{4}<g_{100}(4,l)\leq\binom{98}{4}+\binom{96}{3}+\binom{94}{2}+\binom{92}{1}+\binom{90}{0}=3,759,624$. Finally, for $l=96$, Theorem 2 gives $3,464,840=\binom{97}{4}<g_{100}(4,96)\leq\binom{97}{4}+\binom{95}{3}+\binom{93}{2}+\binom{91}{1}+\binom{89}{0}=3,607,625$.

From these and other similar tables we see how $g_n(m,l)$ decreases as l increases from m to $n-m$. Namely, we observe that the decrease is largest from level m to level $m+1$ and from level $n-m-1$ to level $n-m$, quite modest between level $m+2$ and level $2m$, and that, rather strikingly, $g_n(m,l)$ is constant between levels $l=2m$ and $l=n-m-1$. In fact, we have the following conjectures.

Conjecture 5.11. For $n\gg m$ we have

1. $g_n(m,l)=\binom{n}{m}-\binom{n}{m-1}$ for every $l=2m,2m+1,\dots,n-m-1$, and
2. $g_n(m,n-m)=\binom{n-1}{m}-\binom{n-1}{m-1}$.

Note that the m-binomial representation of $\binom{n}{m}-\binom{n}{m-1}$ starts with $\binom{n-2}{m}+\binom{n-4}{m-1}$ (and $\binom{n-1}{m}-\binom{n-1}{m-1}$ starts with $\binom{n-3}{m}+\binom{n-5}{m-1}$) when $n\gg m$ (cf. Theorem 2). Corollary 3 proves our conjectures for $m=1$, and Theorem 2 establishes that Conjecture 11 provides an upper bound for $m=2$ since $\binom{n}{2}-\binom{n}{1}=\binom{n-2}{2}+\binom{n-3}{1}$ (and $\binom{n-1}{2}-\binom{n-1}{1}=\binom{n-3}{2}+\binom{n-4}{1}$).

According to Conjecture 11, we have $g_n(m,n-m)=g_{n-1}(m,n-m-2)=g_{n-1}(m,l)$ for $n\gg m$ and $2m\leq l\leq n-m-2$. Proposition 7 establishes $g_n(m,n-m)\leq g_{n-1}(m,l)$ for $l<n/2$, while Proposition 10 proves the other direction for $n/2-1<l\leq n-m-2$ (yielding equality when $l=(n-1)/2$).

Note. In a subsequent paper [2], the above conjectures have been somewhat refined and related to other conjectures about the width of cutsets in the truncated Boolean lattice.

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Béla Bajnok

Gettysburg College,
Mathematics, Box 402
Gettysburg, PA 17325, USA
bbajnok@gettysburg.edu

Shahriar Shahriari

Pomona College
Claremont, CA 91711 USA
sshahriari@pomona.edu